

**On Euler—Jaczewski sequence  
and Remmert—Van de Ven problem for toric varieties**

*Dedicated to the memory of Krzysztof Jaczewski*

Gianluca Occhetta and Jarosław A. Wiśniewski

**Introduction.**

In [RV] Remmert and Van de Ven posed a problem concerning holomorphic maps from a complex projective space onto a smooth variety of the same dimension: the question was whether the target variety has to be the projective space as well. The problem has a positive answer provided by Lazarsfeld, [La].

**Theorem.** [Lazarsfeld] *Suppose that  $Y$  is a smooth projective variety of positive dimension. If  $\varphi : \mathbf{P}^n \rightarrow Y$  is a surjective morphism then  $Y \simeq \mathbf{P}^n$ .*

Lazarsfeld's proof depends on a (somewhat technical) characterization of the projective space, by Mori, which was obtained as a by-product of his proof of Frankel-Hartshorne conjecture, [Mo]. We will explain the result in the section on rational curves.

Following Lazarsfeld, questions were raised concerning possible extensions of Remmert – Van de Ven problem for a broader class of varieties. That is, given an  $X$  from a class of varieties and a morphism  $\varphi : X \rightarrow Y$  onto a smooth projective variety (with possibly additional assumptions on  $\varphi$  and  $Y$ , like  $\rho(Y) = 1$ , where  $\rho$  denotes the Picard number, or the rank of Neron-Severi group of the variety), one would like to deduce the structure of  $Y$ , preferably to claim that  $Y \simeq \mathbf{P}^n$ , unless  $\varphi$  is an isomorphism. In particular, the following cases have been considered:  $X$  is a smooth quadric, see [PS] and [CS],  $X$  is an irreducible symmetric Hermitian space, [Ts],  $X$  is rational homogeneous with  $\rho(X) = 1$ , [HM] and  $X$  is a Fano 3-fold, [Am] and [Sc].

The main result of the present note is the following.

**Theorem 1.** *Suppose that  $X$  is a complete toric variety and  $Y$  a smooth projective variety with  $\rho(Y) = 1$ . If  $\varphi : X \rightarrow Y$  is a surjective morphism then  $Y \simeq \mathbf{P}^n$ .*

The proof of the main result follows the line of Lazarsfeld's argument, in particular we apply Mori's ideas of considering families of rational curves. A new ingredient of the proof is Euler-Jaczewski sequence for toric varieties which we explain in the subsequent section. As an application we obtain a result on varieties admitting two projective bundle structures (Theorem 2 in the last section of the paper).

In the present paper all varieties are defined over an algebraically closed field of characteristic zero.

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## Toric varieties.

For generalities on toric varieties we refer the reader to [Fu] or [Od]. Let  $X = X_\Sigma$  be a toric variety defined by a fan  $\Sigma$  in the space  $\mathbf{N}_\mathbf{R}$ , with  $\mathbf{N}$  denoting the lattice of 1-parameter subgroups of the big torus  $\mathbf{T} \simeq (\mathbf{C}^*)^n$  and  $\mathbf{M}$  denoting its dual. By  $\Sigma^1 = \{\rho\}$  let us denote the set of rays (1-dimensional cones) in the fan  $\Sigma$ . The equivariant divisor in  $X$  (the closure of a codimension 1 orbit of the action of  $\mathbf{T}$ ) associated to a ray  $\rho \in \Sigma^1$  we shall denote by  $D_\rho$ . The variety  $X$  is decomposed into the union of the open orbit of  $\mathbf{T}$  and the divisor  $\bigcup_{\rho \in \Sigma^1} D_\rho$ .

Let us recall the following general fact due to Blanchard, [Bl].

**Theorem.** [Blanchard] *Let  $X$  be a normal complete variety and  $G$  be a connected algebraic group acting on  $X$  such that the induced action on  $\text{Pic}(X)$  is trivial (assume, for example, that  $H^1(X, \mathcal{O}_X) = 0$ ). Suppose that  $\varphi : X \rightarrow Y$  is a morphism to a projective normal variety with connected fibers, so that  $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$ . Then there exists an action of  $G$  on  $Y$  such that  $\varphi$  is equivariant.*

**Proof.** Let  $L = \varphi^* L_Y$ , where  $L_Y$  is an ample line bundle over  $Y$ .

We have  $X = \text{Proj}_X(\bigoplus_m L^{\otimes m})$  and  $Y = \text{Proj}(\bigoplus_m H^0(X, L^{\otimes m}))$  and  $\varphi : X \rightarrow Y$  is induced by the evaluation morphism  $H^0(X, L^{\otimes m}) \rightarrow L^{\otimes m}$ . The natural action of  $G$  on the graded ring of sections  $\bigoplus_m H^0(X, L^{\otimes m})$  is clearly compatible with the evaluation.

**Corollary 1.** *Let  $X$  be a toric variety,  $\varphi : X \rightarrow Y$  a morphism to a projective variety  $Y$  and let  $X \xrightarrow{f_0} X' \xrightarrow{f_1} Y$  be the Stein factorization of  $\varphi$ . Then  $X'$  is a toric variety.*

**Proof.** By Blanchard's theorem the big torus of  $X$  acts on  $Y$  with an open orbit. Thus the quotient of the big torus of  $X$  by the isotropy of a general point of  $Y$  is a torus and it acts on  $Y$  with an open orbit, hence  $Y$  is toric by [Od], Theorem 1.5.

**Euler-Jaczewski sequence.** Let  $X$  be a complete algebraic variety, let  $H = H^1(X, \Omega_X)$  and  $\mathcal{H}$  be the sheaf  $H \otimes \mathcal{O}_X$ .

**Definition.** The short exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{P}_X^\vee \longrightarrow \mathcal{H} \longrightarrow 0$$

corresponding to  $\text{Id}_H \in \text{Hom}(H, H) = \text{Ext}^1(\mathcal{H}, \Omega_X)$  will be called the Euler-Jaczewski sequence of the variety  $X$  whereas the sheaf  $\mathcal{P}_X$  will be called the potential sheaf of  $X$ .

We have the following characterization of toric varieties in terms of the Euler-Jaczewski sequence, [Ja, 3.1]:

**Theorem.** [Jaczewski] *A smooth complete and connected variety  $X$  is a toric variety if and only if there exists an effective divisor  $D = \bigcup_i D_i$  with normal crossing such that*

$$\mathcal{P}_X = \bigoplus_{i \in I} \mathcal{O}_X(D_i)$$

where  $D_i$  are the irreducible components of the divisor  $D$ . The divisors  $D_i$  are then the closures of the codimension one orbits of the torus.

The existence of a generalized Euler sequence on a smooth toric variety was discovered by Batyrev and Melnikov in [BM]. The characterization of toric varieties in terms of this sequence was proved later by Jaczewski, [Ja], who apparently was not aware of the Batyrev and Melnikov's work.

On a smooth toric variety we have a short exact sequence:

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathrm{Div}^T X \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$

where  $\mathrm{Div}^T X$  denotes the group of  $\mathbf{T}$  equivariant divisors  $\sum_{\rho} a_{\rho} D_{\rho}$ . The first map associates to a character its divisor of poles and zeroes while the second map associates to a  $\mathbf{T}$ -equivariant divisor its class in  $\mathrm{Pic}(X)$ .

Dualizing the above sequence and tensoring it with  $\mathbf{C}$  we obtain

$$0 \longrightarrow N_1(X)_{\mathbf{C}} \longrightarrow \bigoplus_{\rho \in \Sigma^1} \mathbf{C}[\rho] \longrightarrow \mathbf{N}_{\mathbf{C}} \longrightarrow 0$$

where  $N_1(X)_{\mathbf{C}}$  is the space of 1-cycles on  $X$ .

The maps in the sequence are defined as follows:  $N_1(X)_{\mathbf{C}} \ni Z \longrightarrow \sum_{\rho \in \Sigma^1} (Z \cdot D_{\rho}) \cdot [\rho]$  and  $\sum_{\rho} a_{\rho} \cdot [\rho] \longrightarrow \sum_{\rho} a_{\rho} \cdot e_{\rho}$ , where  $e_{\rho}$  is the generator of the semigroup  $\mathbf{N} \cap \rho$ .

**Lemma 1.** *Let  $X = X_{\Sigma}$  be a toric variety as above. Consider  $\varphi : X \rightarrow Y$  a surjective and generically finite morphism. By  $J(\varphi)$  let us denote the subset of  $\Sigma^1$  corresponding to divisors  $D_{\rho}$  which are mapped to divisors in  $Y$ , that is  $J(\varphi) = \{\rho \in \Sigma^1 \mid \varphi_* D_{\rho} \neq 0\}$ . Then the restriction of the above map  $\bigoplus_{\rho \in J} \mathbf{C}[\rho] \longrightarrow \mathbf{N}_{\mathbf{C}}$  is surjective.*

**Proof.** Consider the Stein factorization of  $\varphi : X \rightarrow Y$ . By corollary 1 the connected part of this factorization is a birational toric morphism  $\varphi_0 : X_{\Sigma} \rightarrow X_{\Sigma'}$ , where the fan  $\Sigma$  is a subdivision of a fan  $\Sigma'$  and the rays of  $\Sigma'$  are exactly the rays corresponding to the divisors  $D_{\rho}$ , with  $\rho \in J(\varphi)$ . Since the fan  $\Sigma'$  is complete, its rays span  $\mathbf{N}_{\mathbf{C}}$  and we are done.

Note that, for a toric variety, we have  $H^1(X, \Omega_X) = \mathrm{Pic}(X)_{\mathbf{C}}$ . In particular  $N_1(X)_{\mathbf{C}} \otimes \mathcal{O}_X = \mathcal{H}^{\vee}$ ; denoting by  $\mathcal{O}_X[\rho]$  the sheaf  $\mathbf{C}[\rho] \otimes \mathcal{O}_X$  and by  $\mathcal{N}$  the sheaf  $\mathbf{N}_{\mathbf{C}} \otimes \mathcal{O}_X$  we have a commutative diagram of sheaves over  $X$  with exact rows and columns, which contains both the Euler-Jaczewski sequence and the sequence we have just discussed (see [Ja], diagram

(7) on page 238):

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{H}^\vee & \longrightarrow & \bigoplus \mathcal{O}_X[\rho] & \longrightarrow & \mathcal{N} \longrightarrow 0 \\
& & \parallel & & \downarrow s & & \downarrow ev \\
0 & \longrightarrow & \mathcal{H}^\vee & \longrightarrow & \bigoplus \mathcal{O}_X(D_\rho) & \longrightarrow & TX \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & \bigoplus \mathcal{O}_{D_\rho}(D_\rho) = \bigoplus \mathcal{O}_{D_\rho}(D_\rho) & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array} \quad (*)$$

Here  $s = (s_\rho)$ , where  $s_\rho$  is the section of  $\mathcal{O}_X(D_\rho)$  vanishing along  $D_\rho$ , and  $ev$  associates to a 1-parameter group  $\gamma \in \mathbf{N}$  its tangent field.

### Families of curves.

Let  $Y$  be a smooth projective variety, and  $y \in Y$  a point. We consider schemes  $\text{Hom}(\mathbf{P}^1, Y)$ , parametrizing morphisms from  $\mathbf{P}^1$  to  $Y$ , and  $\text{Hom}(\mathbf{P}^1, Y; 0 \rightarrow y)$ , parametrizing morphisms sending  $0 \in \mathbf{P}^1$  to  $y \in Y$ . Let  $V \subset \text{Hom}(\mathbf{P}^1, Y)$  be a closed irreducible subvariety; we will call  $V$ , by abuse, a family of rational curves on  $Y$  and we will denote by  $V_y$  the variety  $V \cap \text{Hom}(\mathbf{P}^1, Y; 0 \rightarrow y)$ . If the evaluation  $F : \mathbf{P}^1 \times \text{Hom}(\mathbf{P}^1, Y) \rightarrow Y$  is a dominating morphism then we will call  $V$  a dominating family of rational curves. Suppose that  $\rho(Y) = 1$ . Then among all dominating families of rational curves on  $Y$  (if there exists any) we can choose a family  $\mathcal{V}$  parametrizing curves of minimal degree with respect to a chosen ample divisor on  $Y$ ; we will call such a family a minimal dominating family of rational curves on  $Y$ .

Let us note that in case of Theorem 1, that is when  $Y$  is dominated by a toric variety and  $\rho(Y) = 1$ , there exists a minimal dominating family  $\mathcal{V}$  of rational curves for  $Y$ . For this family we will use the following version of Mori's result, [Mo].

**Theorem.** [Mori] *Assume that  $Y$  is a smooth projective variety such that  $\rho(Y) = 1$ . Let  $\mathcal{V} \subset \text{Hom}(\mathbf{P}^1, Y)$  be a minimal dominating family of rational curves on  $Y$ . If for a general point  $y \in Y$  and for any  $f \in \mathcal{V}_y$  the pull-back  $f^*TY$  is ample then  $Y \simeq \mathbf{P}^n$ .*

For the proof of Theorem 1 we will need to understand properties of the minimal dominating family of rational curves on  $Y$  from the point of view of properties of the dominating variety  $X \rightarrow Y$ . The key is the following technical observation which we prove in a more general setup:

**Lemma 2.** *Let  $\varphi : X \rightarrow Y$  be a surjective morphism of smooth irreducible projective varieties. Assume that  $\mathcal{M}$  is a dominating family of curves of  $Y$ , that is, there exists a*

variety  $\mathcal{C}_Y$  with morphisms  $p : \mathcal{C}_Y \longrightarrow \mathcal{M}$  and  $q : \mathcal{C}_Y \longrightarrow Y$ , the latter morphism is dominating, such that all fibers of  $p$  are 1-dimensional and mapped via  $q$  to curves in  $Y$ . Let  $\mathcal{C}_X$  be an irreducible component of the fiber product  $X \times_Y \mathcal{C}_Y$  which dominates  $\mathcal{C}_Y$ ; by  $\bar{q}$  and  $\bar{\varphi}$  let us denote morphisms of  $\mathcal{C}_X$  to  $X$  and  $\mathcal{C}_Y$ , respectively. Suppose that  $D$  is an irreducible effective Cartier divisor on  $X$  such that  $\varphi_* D$  is an ample Cartier divisor on  $Y$ . Then, for every  $m \in \mathcal{M}$ , the pull-back  $\bar{p}^* D$  is non zero on every connected component of  $(p \circ \bar{\varphi})^{-1}(m)$ .

**Proof.** Let  $\mathcal{C}_X \xrightarrow{p_0} \mathcal{M}_0 \xrightarrow{p_1} \mathcal{M}$  be the Stein factorization of the map  $p \circ \bar{\varphi} : \mathcal{C}_X \longrightarrow \mathcal{M}$ , so that we the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{M}_0 & & \\
 & \nearrow p_0 & & \searrow p_1 & \\
 \mathcal{C}_X & \xrightarrow{\quad \bar{\varphi} \quad} & \mathcal{C}_Y & \xrightarrow{\quad p \quad} & \mathcal{M} \\
 \bar{q} \downarrow & & q \downarrow & & \\
 X & \xrightarrow{\quad \varphi \quad} & Y & & 
 \end{array}$$

By our assumptions  $p_1 \circ p_0 : \bar{q}^* D \rightarrow \mathcal{M}$  dominates  $\mathcal{M}$ , so its image in  $\mathcal{M}_0$  is an irreducible subvariety of maximal dimension, hence, since  $\mathcal{M}_0$  is irreducible, it coincides with  $\mathcal{M}_0$ .

### Proof of Theorem 1.

First of all, by toric Chow's Lemma we can assume that  $X$  is projective. Then, in view of Corollary 1, considering the Stein factorization of  $\varphi$ , we can assume that  $X$  and  $Y$  are of the same dimension. Finally, by taking a desingularization of  $X$ , we can assume that  $X$  is smooth.

Let  $\mathcal{V}$  be a minimal dominating family of rational curves on  $Y$ . Let  $y \in Y$  be a general point which is not contained in the branch locus of  $\varphi$  nor it is contained in the image of  $\varphi(\bigcup D_\rho)$ . Let  $f : \mathbf{P}^1 \rightarrow Y$  be a curve in  $\mathcal{V}$  passing through  $y$ ; the pullback of the tangent bundle splits into a sum of line bundles:  $f^*TY \simeq \bigoplus \mathcal{O}_{\mathbf{P}^1}(a_l)$ . By Mori's theorem explained above we shall be done if all numbers  $a_l$  are positive.

Suppose, by contradiction that  $a_{\bar{l}} \leq 0$  for some  $\bar{l}$ ; in this case we have a surjection  $f^*TY \longrightarrow \mathcal{O}_{\mathbf{P}^1}(a_{\bar{l}})$  and hence a surjection  $f^*TY \longrightarrow \mathcal{O}_{\mathbf{P}^1}$ . By Lemma 1 and the commutativity of diagram (\*) the map  $\bigoplus_{\rho \in J} \mathcal{O}_X(D_\rho) \longrightarrow TX$  is generically surjective. Since we assume that  $\varphi$  is generically finite, we get a generically surjective map  $\bigoplus_{\rho \in J} \mathcal{O}(D_\rho) \rightarrow \varphi^*TY$  which, by the choice of  $y$ , is surjective over  $\varphi^{-1}(y)$ .

Let us take  $\mathcal{C}_Y = \mathcal{V} \times \mathbf{P}^1$  with  $p : \mathcal{C}_Y \longrightarrow \mathcal{V}$  the projection and  $q : \mathcal{C}_Y \longrightarrow Y$  the evaluation. Now consider the situation discussed in Lemma 2

$$\begin{array}{ccccc}
 \mathcal{C}_X & \xrightarrow{\quad \bar{\varphi} \quad} & \mathcal{C}_Y & \xrightarrow{\quad p \quad} & \mathcal{V} \\
 \bar{q} \downarrow & & q \downarrow & & \\
 X & \xrightarrow{\quad \varphi \quad} & Y & & 
 \end{array}$$

Let  $X_f$  be a connected component of  $(p \circ \bar{\varphi})^{-1}(f)$ . We obtain a generically surjective map  $\bar{q}^*(\bigoplus_{\rho \in J} \mathcal{O}(D_\rho))|_{X_f} \longrightarrow ((\bar{q}^* \circ \varphi^*)TY)|_{X_f} = (\bar{\varphi}^*(f^*TY))|_{X_f}$  and therefore we have a generically surjective map  $\bar{q}^*(\bigoplus_{\rho \in J} \mathcal{O}(D_\rho))|_{X_f} \longrightarrow \mathcal{O}_{X_f}$ .

Thus there exists a non-zero section in

$$H^0(X_f, \bar{q}^*(\bigoplus_{\rho \in J} \mathcal{O}(-D_\rho))) = \bigoplus_{\rho \in J} H^0(X_f, \bar{q}^* \mathcal{O}(-D_\rho)),$$

but this is impossible, since  $X_f$  is connected and for any  $\rho \in J$  the divisor  $\bar{q}^* D_\rho$  is effective non-zero on  $X_f$  by Lemma 2.

### An application.

Let  $X$  be a smooth variety endowed with two different  $\mathbf{P}$ -bundle structures  $\varphi : X \rightarrow Y$  and  $\psi : X \rightarrow Z$ . Since fibers of different extremal ray contractions can meet only in points we have  $\dim X \geq \dim Y + \dim Z$ ; an easy corollary of Lazarsfeld's theorem is that we have equality if and only if  $X = \mathbf{P}^r \times \mathbf{P}^s$ .

Using Theorem 1, we are able to describe the next case; we have the following

**Theorem 2.** *Let  $X$  be a smooth projective variety of dimension  $n$ , endowed with two different  $\mathbf{P}$ -bundle structures  $\varphi : X \rightarrow Y$  and  $\psi : X \rightarrow Z$  such that  $\dim Y + \dim Z = n + 1$ . Then either  $n = 2m - 1$ ,  $Y = Z = \mathbf{P}^m$  and  $X = \mathbf{P}(T\mathbf{P}^m)$  or  $Y$  and  $Z$  have a  $\mathbf{P}$ -bundle structure over a smooth curve  $C$  and  $X = Y \times_C Z$ .*

**Proof.** Let  $[l_\varphi]$  and  $[l_\psi]$  be the numerical equivalence classes of lines in the fibers of  $\varphi$  and  $\psi$ . Let  $x$  be a point of  $X$  and let  $B_x$  be the set of points of  $X$  that can be joined to  $x$  by a chain of rational curves whose numerical class is either  $[l_\varphi]$  or  $[l_\psi]$ .

If there exists a point  $x$  such that  $B_x = X$  then, by [Ko, IV.3.13.3] the Picard number of  $X$  is two, hence  $Y$  and  $Z$  are Fano varieties of Picard number one. We also note that every pair of points of  $Z$  is connected by a chain of rational curves whose members are images of rational curves in  $X$  whose numerical class is  $[l_\varphi]$ .

Let  $F_t$  and  $F_w$ , with  $t, w \in Z$ , be two fibers of  $\psi$  such that  $\varphi(F_t) \neq \varphi(F_w)$ . Since points  $t$  and  $w$  are joined by a chain of rational curves as above thus there exists a rational curve  $\Gamma$  on  $Z$  such that  $\varphi : \psi^{-1}(\Gamma) \rightarrow Y$  is dominant.

Let  $\nu : \mathbf{P}^1 \rightarrow Z$  be the normalization of  $\Gamma$  and let  $X_\Gamma \rightarrow \mathbf{P}^1$  be the pull-back of the projective bundle, that is  $X_\Gamma = \mathbf{P}^1 \times_\Gamma X$ . The composition of the induced map  $\bar{\nu} : X_\Gamma \rightarrow X$  with  $\varphi$  is a surjective morphism  $\varphi \circ \bar{\nu} : X_\Gamma \rightarrow Y$ . The variety  $X_\Gamma$  is a  $\mathbf{P}$ -bundle on  $\mathbf{P}^1$ , hence a toric variety, and  $Y$  is a smooth variety with  $\rho(Y) = 1$ , hence Theorem 1 applies to give  $Y \simeq \mathbf{P}^{\dim Y}$ ; in the same way we also get  $Z \simeq \mathbf{P}^{\dim Z}$ .

We conclude this case by the main theorem of [Sa], that is we get  $\dim Y = \dim Z = m$  and  $X \simeq \mathbf{P}(T\mathbf{P}^m)$ .

If  $B_x$  is a divisor for every  $x \in X$  then, for general  $(x_1, x_2)$  in  $X \times X$ , the divisors  $B_{x_1}$  and  $B_{x_2}$  are disjoint and because they are numerically equivalent we have  $B_x^2 \equiv 0$ . We claim that there exists a regular morphism  $p : X \rightarrow C$  with connected fibers, onto a smooth curve  $C$ , which contracts all divisors  $B_x$ . If  $H^1(X, \mathcal{O}_X) \neq 0$  then  $p$  is obtained from the Albanese map of  $X$  (note that  $B_x$  are rationally connected hence contracted by

Albanese). If  $H^1(X, \mathcal{O}_X) = 0$  then divisors  $B_x$  are linearly equivalent, the linear system  $|B_x|$  is base point free and defines  $p$ . By construction  $p$  factors through  $\varphi$  and  $\psi$  to produce  $p_Y : Y \rightarrow C$  and  $p_Z : Z \rightarrow C$ . Moreover, by construction  $\rho(X/C) = 2$  hence  $\rho(Y/C) = 1$  and  $\rho(Z/C) = 1$ .

A general  $B_x$  is smooth with two projective bundle structures hence, by what we have said at the beginning of this section, it is a product of projective spaces. Thus a general fiber of  $p_X$  as well as  $p_Z$  is a projective space and over an open Zariski  $U$  subset of  $C$  both morphisms are projective bundles. By taking the closure in  $Y$  and  $Z$  of a hyperplane section of  $p_Y$  and  $p_Z$ , respectively, defined over the open set  $U$  we get a global relative hyperplane section divisor (we use  $\rho(Z/C) = \rho(Y/C) = 1$ ) hence  $p_Y$  and  $p_Z$  are projective bundles globally. The conclusion  $X = Y \times_C Z$  is immediate.

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Gianluca Occhetta  
Dipartimento di Matematica “F. Enriques”  
Università degli Studi di Milano  
Via Saldini, 50  
I-20133 Milano, Italy  
occhetta@mat.unimi.it

Jarosław A. Wiśniewski  
Instytut Matematyki  
Uniwersytet Warszawski  
Banacha 2  
PL-02097 Warszawa, Poland  
jarekw@mimuw.edu.pl